SLAM EKF: the insertion Jacobian

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The EKF formulation for mapping and SLAM requires that the state vector is extended every time a new landmark is discovered. For the map-making case the state vector is

$$\boldsymbol{x} = (\boldsymbol{x}_m) \in \mathbb{R}^{n \times 1}$$

where $\boldsymbol{x}_m \in \mathbb{R}^{2m \times 1}$ and m is the number of landmarks so $n = 2m$. For the SLAM case the state vector is

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_v \\ \boldsymbol{x}_m \end{pmatrix} \in \mathbb{R}^{n \times 1}$$

where $\boldsymbol{x}_v \in \mathbb{R}^{3 \times 1}$ is the vehicle configuration so $n = 2m + 3$.

1 Mapping case

The state vector is extended by the function

$$\boldsymbol{y}(\boldsymbol{x}, \boldsymbol{z}) = \begin{matrix} \text{n} \\ 2 \\ \\ 1 \end{matrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{g}(\boldsymbol{x}_v, \boldsymbol{z}) \end{pmatrix}$$

which simply the concatenation the two vectors, and where $\boldsymbol{z}$ is the observation. The Jacobians of this function are

$$\mathbf{Y}_x = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} = \begin{pmatrix} \partial \boldsymbol{x} / \partial \boldsymbol{x} \\ \partial \boldsymbol{g} / \partial \boldsymbol{x} \end{pmatrix} = \begin{matrix} \text{n} \\ 2 \\ \\ \text{n} \end{matrix} \begin{pmatrix} \mathbf{1}_{n \times n} \\ \mathbf{0}_{2 \times n} \end{pmatrix} \in \mathbb{R}^{n+2 \times n}$$

$$\mathbf{Y}_z = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{z}} = \begin{pmatrix} \partial \boldsymbol{x} / \partial \boldsymbol{z} \\ \partial \boldsymbol{g} / \partial \boldsymbol{z} \end{pmatrix} = \begin{matrix} \text{n} \\ 2 \\ \\ 2 \end{matrix} \begin{pmatrix} \mathbf{0}_{n \times 2} \\ \mathbf{G}_z \end{pmatrix} \in \mathbb{R}^{n \times 2}$$

The zero element of $\mathbf{Y}_x$ is because the state vector comprises only landmark coordinates and $g(\cdot)$ is a function of vehicle configuration which is not part of the state vector.
The linearized covariance update, as discussed in Setion H.2, is

$$\mathbf{Y}_x \mathbf{P} \mathbf{Y}_x^T + \mathbf{Y}_z \mathbf{W} \mathbf{Y}_z^T$$

and substituting the Jacobians we can write

$$= \begin{matrix} \text{n} \\ 2 \end{matrix} \begin{pmatrix} \mathbf{P} & \mathbf{0}_{n\times 2} \\ \mathbf{0}_{2\times n} & \mathbf{0}_{2\times 2} \end{pmatrix} \begin{matrix} \\ \text{n} \end{matrix} \begin{matrix} \\ 2 \end{matrix} + \begin{matrix} \text{n} \\ 2 \end{matrix} \begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 2} \\ \mathbf{0}_{2\times n} & \mathbf{G}_z\mathbf{W}\mathbf{G}_z^T \end{pmatrix} \begin{matrix} \\ \text{n} \end{matrix} \begin{matrix} \\ 2 \end{matrix} \tag{1}$$

$$= \begin{pmatrix} \mathbf{P} & \mathbf{0}_{n\times 2} \\ \mathbf{0}_{2\times n} & \mathbf{G}_z\mathbf{W}\mathbf{G}_z^T \end{pmatrix} \tag{2}$$

which, using the procedure of Appendix A, can be shown to be equivalent to the quadratic expression

$$\begin{pmatrix} \mathbf{1}_{n\times n} & \mathbf{0}_{n\times 2} \\ \mathbf{0}_{2\times n} & \mathbf{G}_z \end{pmatrix} \begin{pmatrix} \mathbf{P} & \mathbf{0}_{n\times 2} \\ \mathbf{0}_{2\times n} & \mathbf{W} \end{pmatrix} \begin{pmatrix} \mathbf{1}_{n\times n} & \mathbf{0}_{n\times 2} \\ \mathbf{0}_{2\times n} & \mathbf{G}_z \end{pmatrix}^T$$

where

$$\begin{matrix} \text{n} \\ 2 \end{matrix} \begin{pmatrix} \mathbf{1}_{n\times n} & \mathbf{0}_{n\times 2} \\ \mathbf{0}_{2\times n} & \mathbf{G}_z \end{pmatrix} \begin{matrix} \\ \text{n} \end{matrix} \begin{matrix} \\ 2 \end{matrix}$$

is the insertion Jacobian.

2 SLAM case

The state vector is extended by the function

$$\boldsymbol{y}(\boldsymbol{x},\, \boldsymbol{z}) = \begin{matrix} \text{n} \\ 2 \end{matrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{g}(\boldsymbol{x}_v,\, \boldsymbol{z}) \end{pmatrix} \begin{matrix} \\ 1 \end{matrix}$$

which simply the concatenation the two vectors, and where $\boldsymbol{z}$ is the observation. Recall that the state vector contains the vehicle configuration

$$\boldsymbol{y}(\boldsymbol{x},\, \boldsymbol{z}) = \begin{matrix} 3 \\ 2\text{m} \\ 2 \end{matrix} \begin{pmatrix} \boldsymbol{x}_v \\ \boldsymbol{x}_m \\ \boldsymbol{g}(\boldsymbol{x}_v,\, \boldsymbol{z}) \end{pmatrix} \begin{matrix} \\ \\ 1 \end{matrix}$$

The Jacobians of this function are

$$\mathbf{Y}_x = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} = \begin{pmatrix} \partial \boldsymbol{x}_v/\partial \boldsymbol{x}_v & \partial \boldsymbol{x}_v/\partial \boldsymbol{x}_m \\ \partial \boldsymbol{x}_m/\partial \boldsymbol{x}_v & \partial \boldsymbol{x}_m/\partial \boldsymbol{x}_m \\ \partial \boldsymbol{g}/\partial \boldsymbol{x}_v & \partial \boldsymbol{g}/\partial \boldsymbol{x}_m \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{1}_{3\times 3} & \mathbf{0}_{3\times 2m} \\ \mathbf{0}_{2m\times 3} & \mathbf{1}_{2m\times 2m} \\ \mathbf{G}_{xv} & \mathbf{0}_{2\times 2m} \end{pmatrix} = \begin{matrix} \text{n} \\ 2 \end{matrix} \begin{pmatrix} \mathbf{1}_{n\times n} \\ \mathbf{G}_x \end{pmatrix} \begin{matrix} \\ \text{n} \end{matrix} \in \mathbb{R}^{n+2\times n}$$

where

$$\mathbf{G}_x = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} \in \mathbb{R}^{2\times n}$$

which accounts for the fact that the first three elements of $\boldsymbol{x}$ are the vehicle configuration.

$$\mathbf{Y}_z = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{z}} = \begin{pmatrix} \partial \boldsymbol{x}/\partial \boldsymbol{z} \\ \partial \boldsymbol{g}/\partial \boldsymbol{z} \end{pmatrix} = \begin{matrix} \text{n} \\ 2 \end{matrix} \overbrace{\begin{pmatrix} \mathbf{0}_{n\times 2} \\ \mathbf{G}_z \end{pmatrix}}^{2} \in \mathbb{R}^{n\times 2}$$

The linearized covariance update, as discussed in Setion H.2, is

$$\mathbf{Y}_x \mathbf{P} \mathbf{Y}_x^T + \mathbf{Y}_z \mathbf{W} \mathbf{Y}_z^T$$

and substituting the Jacobians we can write

$$= \begin{matrix} \text{n} \\ 2 \end{matrix} \overbrace{\begin{pmatrix} \mathbf{P} & \mathbf{P}\mathbf{G}_x^T \\ \mathbf{G}_x\mathbf{P} & \mathbf{G}_x\mathbf{P}\mathbf{G}_x^T \end{pmatrix}}^{\text{n} \quad 2} + \begin{matrix} \text{n} \\ 2 \end{matrix} \overbrace{\begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 2} \\ \mathbf{0}_{2\times n} & \mathbf{G}_z\mathbf{W}\mathbf{G}_z^T \end{pmatrix}}^{\text{n} \quad 2} \tag{3}$$

$$= \begin{matrix} \text{n} \\ 2 \end{matrix} \overbrace{\begin{pmatrix} \mathbf{P} & \mathbf{P}\mathbf{G}_x^T \\ \mathbf{G}_x\mathbf{P} & \mathbf{G}_x\mathbf{P}\mathbf{G}_x^T + \mathbf{G}_z\mathbf{W}\mathbf{G}_z^T \end{pmatrix}}^{\text{n} \quad 2} \tag{4}$$

which, using the procedure of Appendix A, can be shown to be equivalent to the quadratic expression

$$\begin{pmatrix} \mathbf{1}_{n\times n} & \mathbf{0}_{n\times 2} \\ \mathbf{G}_x & \mathbf{G}_z \end{pmatrix} \begin{pmatrix} \mathbf{P} & \mathbf{0}_{n\times 2} \\ \mathbf{0}_{2\times n} & \mathbf{W} \end{pmatrix} \begin{pmatrix} \mathbf{1}_{n\times n} & \mathbf{0}_{n\times 2} \\ \mathbf{G}_x & \mathbf{G}_z \end{pmatrix}^T$$

where

$$\begin{matrix} \text{n} \\ 2 \end{matrix} \overbrace{\begin{pmatrix} \mathbf{1}_{n\times n} & \mathbf{0}_{n\times 2} \\ \mathbf{G}_x & \mathbf{G}_z \end{pmatrix}}^{\text{n} \quad 2}$$

is the insertion Jacobian.

A Factoring the Jacobian

Consider a quadratic expression

$$\mathbf{M} \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{W} \end{pmatrix} \mathbf{M}^T$$

where $\mathbf{M}$ is a general block-structured matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$$

which we can expand as

$$\begin{pmatrix} \mathbf{A}\mathbf{P}\mathbf{A}^T + \mathbf{B}\mathbf{W}\mathbf{B}^T & \mathbf{A}\mathbf{P}\mathbf{C}^T + \mathbf{B}\mathbf{W}\mathbf{D}^T \\ \mathbf{C}\mathbf{P}\mathbf{A}^T + \mathbf{D}\mathbf{W}\mathbf{B}^T & \mathbf{C}\mathbf{P}\mathbf{C}^T + \mathbf{D}\mathbf{W}\mathbf{D}^T \end{pmatrix}$$

Now, given a matrix of the form

$$\begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_z\mathbf{W}\mathbf{G}_z^T \end{pmatrix}$$

we can equate coefficients to determine that $\mathbf{A} \equiv 1$, $\mathbf{B} \equiv 0$, $\mathbf{C} \equiv 0$ and $\mathbf{D} \equiv \mathbf{G}_z$ leading to a factorization as

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_z \end{pmatrix} \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{W} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_z \end{pmatrix}^T$$