

SLAM EKF: the insertion Jacobian

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The EKF formulation for mapping and SLAM requires that the state vector is extended every time a new landmark is discovered. For the map-making case the state vector is

$$\mathbf{x} = (\mathbf{x}_m) \in \mathbb{R}^{n \times 1}$$

where $\mathbf{x}_m \in \mathbb{R}^{2m \times 1}$ and m is the number of landmarks so $n = 2m$. For the SLAM case the state vector is

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_v \\ \mathbf{x}_m \end{pmatrix} \in \mathbb{R}^{n \times 1}$$

where $\mathbf{x}_v \in \mathbb{R}^{3 \times 1}$ is the vehicle configuration so $n = 2m + 3$.

1 Mapping case

The state vector is extended by the function

$$\mathbf{y}(\mathbf{x}, \mathbf{z}) = \begin{matrix} \textcolor{red}{n} \\ \textcolor{red}{2} \\ \textcolor{red}{1} \end{matrix} \begin{pmatrix} \mathbf{x} \\ g(\mathbf{x}_v, \mathbf{z}) \end{pmatrix}$$

which simply the concatenation the two vectors, and where $\mathbf{z}$ is the observation. The Jacobians of this function are

$$\mathbf{Y}_x = \frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{pmatrix} \partial \mathbf{x} / \partial \mathbf{x} \\ \partial g / \partial \mathbf{x} \end{pmatrix} = \begin{matrix} \textcolor{red}{n} \\ \textcolor{red}{2} \\ \textcolor{red}{n} \end{matrix} \begin{pmatrix} \mathbf{1}_{n \times n} \\ \mathbf{0}_{2 \times n} \end{pmatrix} \in \mathbb{R}^{n+2 \times n}$$

$$\mathbf{Y}_z = \frac{\partial \mathbf{y}}{\partial \mathbf{z}} = \begin{pmatrix} \partial \mathbf{x} / \partial \mathbf{z} \\ \partial g / \partial \mathbf{z} \end{pmatrix} = \begin{matrix} \textcolor{red}{n} \\ \textcolor{red}{2} \\ \textcolor{red}{2} \end{matrix} \begin{pmatrix} \mathbf{0}_{n \times 2} \\ \mathbf{G}_z \end{pmatrix} \in \mathbb{R}^{n \times 2}$$

The zero element of $\mathbf{Y}_x$ is because the state vector comprises only landmark coordinates and $g(\cdot)$ is a function of vehicle configuration which is not part of the state vector.

The linearized covariance update, as discussed in Setion H.2, is

$$\mathbf{Y}_x \mathbf{P} \mathbf{Y}_x^T + \mathbf{Y}_z \mathbf{W} \mathbf{Y}_z^T$$

and substituting the Jacobians we can write

$$= \begin{matrix} \text{n} \\ \text{2} \end{matrix} \begin{pmatrix} \mathbf{P} & \mathbf{0}_{n \times 2} \\ \mathbf{0}_{2 \times n} & \mathbf{0}_{2 \times 2} \end{pmatrix} + \begin{matrix} \text{n} \\ \text{2} \end{matrix} \begin{pmatrix} \mathbf{0}_{n \times n} & \mathbf{0}_{n \times 2} \\ \mathbf{0}_{2 \times n} & \mathbf{G}_z \mathbf{W} \mathbf{G}_z^T \end{pmatrix} \quad (1)$$

$$= \begin{pmatrix} \mathbf{P} & \mathbf{0}_{n \times 2} \\ \mathbf{0}_{2 \times n} & \mathbf{G}_z \mathbf{W} \mathbf{G}_z^T \end{pmatrix} \quad (2)$$

which, using the procedure of Appendix A, can be shown to be equivalent to the quadratic expression

$$\begin{pmatrix} \mathbf{1}_{n \times n} & \mathbf{0}_{n \times 2} \\ \mathbf{0}_{2 \times n} & \mathbf{G}_z \end{pmatrix} \begin{pmatrix} \mathbf{P} & \mathbf{0}_{n \times 2} \\ \mathbf{0}_{2 \times n} & \mathbf{W} \end{pmatrix} \begin{pmatrix} \mathbf{1}_{n \times n} & \mathbf{0}_{n \times 2} \\ \mathbf{0}_{2 \times n} & \mathbf{G}_z \end{pmatrix}^T$$

where

$$\begin{matrix} \text{n} \\ \text{2} \end{matrix} \begin{pmatrix} \mathbf{1}_{n \times n} & \mathbf{0}_{n \times 2} \\ \mathbf{0}_{2 \times n} & \mathbf{G}_z \end{pmatrix}$$

is the insertion Jacobian.

2 SLAM case

The state vector is extended by the function

$$\mathbf{y}(\mathbf{x}, \mathbf{z}) = \begin{matrix} \text{n} \\ \text{2} \\ \text{1} \end{matrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{g}(\mathbf{x}_v, \mathbf{z}) \end{pmatrix}$$

which simply the concatenation the two vectors, and where $\mathbf{z}$ is the observation. Recall that the state vector contains the vehicle configuration

$$\mathbf{y}(\mathbf{x}, \mathbf{z}) = \begin{matrix} \text{3} \\ \text{2m} \\ \text{2} \\ \text{1} \end{matrix} \begin{pmatrix} \mathbf{x}_v \\ \mathbf{x}_m \\ \mathbf{g}(\mathbf{x}_v, \mathbf{z}) \end{pmatrix}$$

The Jacobians of this function are

$$\begin{aligned} \mathbf{Y}_x &= \frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{pmatrix} \partial \mathbf{x}_v / \partial \mathbf{x}_v & \partial \mathbf{x}_v / \partial \mathbf{x}_m \\ \partial \mathbf{x}_m / \partial \mathbf{x}_v & \partial \mathbf{x}_m / \partial \mathbf{x}_m \\ \partial \mathbf{g} / \partial \mathbf{x}_v & \partial \mathbf{g} / \partial \mathbf{x}_m \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{1}_{3 \times 3} & \mathbf{0}_{3 \times 2m} \\ \mathbf{0}_{2m \times 3} & \mathbf{1}_{2m \times 2m} \\ \mathbf{G}_{xv} & \mathbf{0}_{2 \times 2m} \end{pmatrix} = \begin{matrix} \text{n} \\ \text{2} \end{matrix} \begin{pmatrix} \mathbf{1}_{n \times n} \\ \mathbf{G}_x \end{pmatrix} \in \mathbb{R}^{n+2 \times n} \end{aligned}$$

where

$$\mathbf{G}_x = \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \in \mathbb{R}^{2 \times n}$$

which accounts for the fact that the first three elements of $\mathbf{x}$ are the vehicle configuration.

$$\mathbf{Y}_z = \frac{\partial \mathbf{y}}{\partial \mathbf{z}} = \begin{pmatrix} \partial \mathbf{x} / \partial \mathbf{z} \\ \partial \mathbf{g} / \partial \mathbf{z} \end{pmatrix} = \begin{matrix} \textcolor{red}{n} \\ \textcolor{red}{2} \end{matrix} \begin{pmatrix} \mathbf{0}_{n \times 2} \\ \mathbf{G}_z \end{pmatrix} \in \mathbb{R}^{n \times 2}$$

The linearized covariance update, as discussed in Section H.2, is

$$\mathbf{Y}_x \mathbf{P} \mathbf{Y}_x^T + \mathbf{Y}_z \mathbf{W} \mathbf{Y}_z^T$$

and substituting the Jacobians we can write

$$= \begin{matrix} \textcolor{red}{n} \\ \textcolor{red}{2} \end{matrix} \begin{pmatrix} \mathbf{P} & \mathbf{P} \mathbf{G}_x^T \\ \mathbf{G}_x \mathbf{P} & \mathbf{G}_x \mathbf{P} \mathbf{G}_x^T \end{pmatrix} + \begin{matrix} \textcolor{red}{n} \\ \textcolor{red}{2} \end{matrix} \begin{pmatrix} \mathbf{0}_{n \times n} & \mathbf{0}_{n \times 2} \\ \mathbf{0}_{2 \times n} & \mathbf{G}_z \mathbf{W} \mathbf{G}_z^T \end{pmatrix} \quad (3)$$

$$= \begin{matrix} \textcolor{red}{n} \\ \textcolor{red}{2} \end{matrix} \begin{pmatrix} \mathbf{P} & \mathbf{P} \mathbf{G}_x^T \\ \mathbf{G}_x \mathbf{P} & \mathbf{G}_x \mathbf{P} \mathbf{G}_x^T + \mathbf{G}_z \mathbf{W} \mathbf{G}_z^T \end{pmatrix} \quad (4)$$

which, using the procedure of Appendix A, can be shown to be equivalent to the quadratic expression

$$\begin{pmatrix} \mathbf{1}_{n \times n} & \mathbf{0}_{n \times 2} \\ \mathbf{G}_x & \mathbf{G}_z \end{pmatrix} \begin{pmatrix} \mathbf{P} & \mathbf{0}_{n \times 2} \\ \mathbf{0}_{2 \times n} & \mathbf{W} \end{pmatrix} \begin{pmatrix} \mathbf{1}_{n \times n} & \mathbf{0}_{n \times 2} \\ \mathbf{G}_x & \mathbf{G}_z \end{pmatrix}^T$$

where

$$\begin{matrix} \textcolor{red}{n} \\ \textcolor{red}{2} \end{matrix} \begin{pmatrix} \mathbf{1}_{n \times n} & \mathbf{0}_{n \times 2} \\ \mathbf{G}_x & \mathbf{G}_z \end{pmatrix}$$

is the insertion Jacobian.

A Factoring the Jacobian

Consider a quadratic expression

$$\mathbf{M} \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{W} \end{pmatrix} \mathbf{M}^T$$

where $\mathbf{M}$ is a general block-structured matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$$

which we can expand as

$$\begin{pmatrix} \mathbf{A} \mathbf{P} \mathbf{A}^T + \mathbf{B} \mathbf{W} \mathbf{B}^T & \mathbf{A} \mathbf{P} \mathbf{C}^T + \mathbf{B} \mathbf{W} \mathbf{D}^T \\ \mathbf{C} \mathbf{P} \mathbf{A}^T + \mathbf{D} \mathbf{W} \mathbf{B}^T & \mathbf{C} \mathbf{P} \mathbf{C}^T + \mathbf{D} \mathbf{W} \mathbf{D}^T \end{pmatrix}$$

Now, given a matrix of the form

$$\begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_z \mathbf{W} \mathbf{G}_z^T \end{pmatrix}$$

we can equate coefficients to determine that $\mathbf{A} \equiv \mathbf{1}$, $\mathbf{B} \equiv \mathbf{0}$, $\mathbf{C} \equiv \mathbf{0}$ and $\mathbf{D} \equiv \mathbf{G}_z$ leading to a factorization as

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_z \end{pmatrix} \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{W} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_z \end{pmatrix}^T$$