Solving trigonometric equations

Peter Corke

September 29, 2026

When solving problems in inverse kinematics we often end up with an equation of the form. There are several ways to solve this and they are discussed below.

Simple derivation

$$a\cos\theta + b\sin\theta = c,\ \ a, b \neq 0 \tag{1}$$

and we note similarity with the form of the sum of angles identity

$$\underbrace{\sin\phi}_{\sim a}\cos\theta + \underbrace{\cos\phi}_{\sim b}\sin\theta \equiv \sin(\phi + \theta)$$

In order to equate coefficients we need to ensure that $\sin^2\phi + \cos^2\phi = 1$ which is true only if $a^2 + b^2 = 1$. In general this will not be the case so we normalize the equation, dividing each side by $d = \sqrt{a^2 + b^2}$ giving

$$a'\cos\theta + b'\sin\theta = c' \tag{2}$$

where $a' = a/d$, $b' = b/d$ and $c' = c/d$. Now we can write

$$\sin\phi = a',\ \cos\phi = b'$$

and solve for ϕ

$$\tan\phi = \frac{a'}{b'} = \frac{a}{b} \in [-2\pi, 2\pi)$$

which should be computed using an `atan2` function.
Next we rewrite (2) as

$$\sin(\phi + \theta) = c'$$

and if $|c'| \leq 1$ or $a^2 + b^2 - c^2 > 0$ we can solve for

$$\theta = \sin^{-1} c' - \phi$$

In general, there is a second solution corresponding to the negative solution of the square root $d = -\sqrt{a^2 + b^2}$ leading to

$$\tan\phi = \frac{-a'}{-b'} = \frac{-a}{-b} \in [-2\pi, 2\pi)$$

which puts the solution for ϕ in the diagonally opposite quadrant, and

$$\sin(\phi + \theta) = -c'$$

Since $\sin(-x) = -\sin(x)$ we can write the second solution as

$$\theta = -\sin^{-1} c' - \phi$$

In summary, the two solutions are

$$\begin{aligned} \theta &= \quad \sin^{-1} c' - \phi, \quad \tan\phi = \tfrac{a}{b} \\ \theta &= -\sin^{-1} c' - \phi, \quad \tan\phi = \tfrac{-a}{-b} \end{aligned}$$

We can test this numerically using MATLAB

```
a = 4; b = 5; c = 3;
clear theta

phi = atan2(a, b);
th1 = asin(c/norm([a b])) - phi
```

```
th1 = -0.1871
```

```
phi = atan2(-a, -b);
th2 = -asin(c/norm([a b])) - phi;
theta = [th1 th2];
a*cos(theta) + b*sin(theta) - c
```

```
ans = 1x2
1.0e+-15 *
   -0.8882    -0.8882
```

which indicates solutions equal to zero up to machine precision.

Other forms

Another commonly given solutions of this equation include

$$\theta = \tan^{-1} \frac{c}{\pm\sqrt{a^2 + b^2 - c^2}} - \tan^{-1} \frac{a}{b}$$

and

$$\theta = \tan^{-1} \frac{bc \pm ad}{ac \mp bd}$$

which requires just a single arc-tangent operation. See the discussion at
https://math.stackexchange.com/questions/213545/solving-trigonometric-equations-of-

The Weierstrass transformation

A well known way to convert trigonometric equations to algebraic equations is with the Weierstrass transformation[1] which is familiar as one of the half-angle identities

$$\sin\theta = \frac{2h}{1+h^2}, \quad \cos\theta = \frac{1-h^2}{1+h^2}$$

where $h = \tan\frac{\theta}{2}$. The problem (1) can be rewritten as

$$\frac{2bh}{h^2+1} - \frac{a(h^2-1)}{h^2+1} = c$$

which can be expressed as a quadratic in h

$$(a+c)h^2 - 2bh + a + c = 0$$

which we can solve as

$$h = \begin{pmatrix} \frac{b+\sqrt{a^2+b^2-c^2}}{a+c} \\ \frac{b-\sqrt{a^2+b^2-c^2}}{a+c} \end{pmatrix} \tag{3}$$

and clearly has the two solutions so long as $a^2+b^2-c^2 > 0$. The transformation has led to the solution in a very straightforward fashion.
Once again, we can test this numerically using MATLAB

```
syms a b c h theta
e = a*cos(theta) + b*sin(theta) == c
```

e =

$$a\cos(\theta) + b\sin(\theta) = c$$

```
e = rewrite(e, 'tan')
```

e =

$$\frac{2b\tan\left(\frac{\theta}{2}\right)}{\tan\left(\frac{\theta}{2}\right)^2+1} - \frac{a\left(\tan\left(\frac{\theta}{2}\right)^2-1\right)}{\tan\left(\frac{\theta}{2}\right)^2+1} = c$$

```
e = subs(e, tan(theta/2), h)
```

e =

$$\frac{2bh}{h^2+1} - \frac{a\left(h^2-1\right)}{h^2+1} = c$$

[1] After the German mathematician Karl Weierstrass (1815-1897).

```
sol = solve(e, h)
```

```
sol =
```

$$\begin{pmatrix} \frac{b+\sqrt{a^2+b^2-c^2}}{a+c} \\ \frac{b-\sqrt{a^2+b^2-c^2}}{a+c} \end{pmatrix}$$

Now lets validate this

```
a = 4; b = 5; c = 3;
theta = 2*atan(eval(sol))
```

```
theta = 2x1
    1.9792
   -0.1871
```

```
a*cos(theta) + b*sin(theta) - c
```

```
ans = 2x1
1.0e+-15 *
         0
   -0.8882
```

which again is equal to zero up to machine precision.